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An Equivalence Theorem Concerning Multipliers of Strong Convergence

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1. INTRODUCTION

The purpose of this article is to extend some classical results concerning multipliers of uniform convergence for one-dimensional trigonometric series to the setting of abstract Fourier expansions in Banach spaces. In fact, the starting point for the present investigations may be considered to be the following well-known equivalence assertion.

Let $C_{2\pi}$ be the space of 2π-periodic continuous functions f with norm $\|f\|_{C_{2\pi}} := \max_{-\pi \leqslant u \leqslant \pi} |f(u)|$, and let $V \subset C_{2\pi}$ be any subset. Setting

$$f(t) \sim \sum_{k=-\infty}^{\infty} f^{\wedge}(k) e^{ikt}, \qquad f^{\wedge}(k) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(u) e^{-iku}\, du, \tag{1.1}$$

$$(C_{2\pi})_0 := \left\{ f \in C_{2\pi};\ \lim_{n \to \infty} \left\| \sum_{k=-n}^{n} f^{\wedge}(k) e^{ikt} - f(t) \right\|_{C_{2\pi}} = 0 \right\}, \tag{1.2}$$

a classical problem asks for properties that an arbitrary sequence $\tau := \{\tau_k\}_{k=-\infty}^{\infty}$ of complex numbers should satisfy such that $f \in V$ always implies the uniform convergence of the series $\sum_{k=-\infty}^{\infty} \tau_k f^{\wedge}(k) e^{ikt}$, thus defining an element $f^{\tau} \in (C_{2\pi})_0$. Such a factor sequence τ is called a multiplier of uniform (or strong) convergence for V, in notation $\tau \in M(V, (C_{2\pi})_0)$. Let

$$(C_{2\pi})_{\omega} := \{ f \in C_{2\pi};\ \omega(f; \delta) = \mathcal{O}(\omega(\delta)),\ \delta \to 0+ \}, \tag{1.3}$$

where $\omega(t)$ is any modulus of continuity (cf. (3.3)), and

$$\omega(f; \delta) := \sup_{|h| \leqslant \delta} \|f(u+h) - f(u)\|_{C_{2\pi}}. \tag{1.4}$$

Among the many contributions devoted to this problem we would like to

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consider the following one, concerned with necessary and sufficient conditions given in terms of the kernel of τ, i.e.,

$$D_n^\tau(t) := \sum_{k=-n}^{n} \tau_k e^{ikt}. \tag{1.5}$$

THEOREM 1.1. *Let* $\tau \in M(C_{2\pi}, C_{2\pi})$ (*cf. Definition* 2.2). *Then* $(n \to \infty)$

$$\omega(1/n) \int_{-\pi}^{\pi} |D_n^\tau(u)|\, du = o(1) \Leftrightarrow \tau \in M((C_{2\pi})_\omega, (C_{2\pi})_0).$$

In this final version the result is due to Teljakovskii [36], who in particular supplied the necessity part of the assertion. This result and many others as well as their methods of proof, however, were subject to a long development involving the work of many mathematicians, in connection with Theorem 1.1 notably that of Tomic [38, 39], Karamata [17], Aljancic [1], Bojanic [2], Katayama [18], Goes [12], Harsiladse [14], DeVore [10]; it was followed by that of Pochuev [29] (for results of a somewhat different nature cf. [15, 31], see also Section 7, and for detailed bibliographical comments [20]). Let us observe that these contributions are exclusively concerned with one-dimensional trigonometric expansions, apart from those of Aljancic, which deal with an arbitrary orthonormal sequence of functions in $C[a, b]$: these are, however, preliminary in comparison with Theorem 1.1, see Proposition 3.1.

This article aims to discuss multipliers of strong convergence within the abstract setting of biorthogonal systems in Banach spaces, thus deriving counterparts of Theorem 1.1 for a fairly general class of orthogonal expansions.

To this end, Section 2 reviews some basic facts concerning multipliers between Banach spaces which are admissible (or regular) with respect to a given orthonormal sequence in some Hilbert space. In view of these concepts—they were already developed in our previous studies (cf. [7, 8, 24, 26, 40])—it is possible to consider one and the same orthonormal system in different Banach spaces. Another important feature is that generalized de La Vallée Poussin (or delayed) means are readily available (cf. Lemma 2.5). In Section 3 we treat sufficient conditions for multipliers of strong convergence, thereby slightly generalizing our results given in [24]. It may be worthwhile to mention that a Jackson inequality of type (3.11) plays a role in some of the assertions. The main aspect, however, is dealt with in Section 4, where necessary conditions for multipliers of strong convergence are studied on the basis of a Bernstein inequality of type (4.4). The results of Sections 3 and 4 then immediately lead to the equivalence assertions of Section 5; they also include corresponding ones for multipliers of uniform boundedness (cf.

(3.13)). In Section 6 we indicate some applications. Here we are by no means systematic. First we show that Theorem 1.1 as well as counterparts of DeVore and Pochuev may indeed be regained in a unified way as an immediate application of Theorem 5.1 (cf. Corollary 6.1, 6.2). Then we consider multipliers of strong convergence in connection with multivariate trigonometric and Jacobi expansions. Rather than to formulate various results explicitly, emphasis is laid upon a careful discussion of the verification of the assumptions needed for the application of the abstract results. Finally, Section 7 outlines some further extensions and results.

The use of a Jackson and a Bernstein inequality to treat sufficiency and necessity, respectively, was already employed by Teljakovskii in his original treatment of Theorem 1.1. In fact, this procedure may be considered in connection with the Butzer–Scherer constructive theory of functions which derives direct approximation theorems from a Jackson-type inequality and inverse ones from a Bernstein-type inequality (cf. [9] and the literature cited there). Therefore it was important even for the development of the concept of admissible Banach spaces that such fundamental inequalities were indeed available in this general frame (cf. [5, 7, 8, 13, 40]). Moreover, the proof of the necessity in the equivalence assertion depends heavily upon the existence of a regularization process such as the de La Vallée Poussin means. In this sense the present generalizations were indeed motivated by the fact that counterparts of such means for regular Banach spaces were already introduced in our previous studies (cf. [24–26]; see also [32]). Teljakovskii's proof of the necessity then proceeds via the construction of some counterexample which in fact may be considered as a suitable modification of the familiar "gliding hump method." Treatment of such problems in an abstract setting as in Section 4 exhibits more clearly its relation to the classical work of, e.g., O. Toeplitz, 1911: so results concerning multipliers of strong convergence may also be understood as "Toeplitz-type theorems" (for orthogonal expansions), a nomenclature suggested by Kojima [19] in his studies on variants of Schur's theorem. In fact, Section 4 may be looked upon as a contribution to the question of how to equip the uniform boundedness principle with orders. In this connection one may also compare the present treatment with the corresponding one of Banach–Steinhaus theorems with orders (cf. [3] and the literature cited there).

2. Admissible Banach Spaces

Let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}_0$, and $\mathbb{N}$ be the set of all complex, real, integral, non-negative integral, and natural numbers, respectively. For some $N \in \mathbb{N}$ the N-fold Cartesean product of $\mathbb{Z}$, $\mathbb{R}$ is denoted by $\mathbb{R}^N$, $\mathbb{R}^N$, respectively. Let $[\![\alpha]\!]$ be the greatest integer less than or equal to $\alpha \geqslant 0$.

Let X, Y be complex Banach spaces, with norm $\|\cdot\|_X$ and dual X^*, for example. Then $[X, Y]$ denotes the Banach space of bounded linear operators of X into Y. In particular, we set $[X, X] := [X]$.

For an arbitrary Hilbert space H with inner product $(\cdot, \cdot)$ let $\{f_k; k \in \mathbb{Z}^N\}$ be an orthonormal sequence, thus $(f_j, f_k) = \delta_{jk}$, $j, k \in \mathbb{Z}^N$, δ_{jk} being Kronecker's symbol. Instead of $\mathbb{Z}^N$ one may use any other denumerable index set, for example, $\mathbb{N}_0$ or $\mathbb{N}$. Let

$$\Pi := \Pi(\{f_k\}) := \left\{p \in H; p = \sum_{\text{finite}} \alpha_k f_k, \alpha_k \in \mathbb{C}\right\} \tag{2.1}$$

be the set of polynomials generated by $\{f_k\}$, and

$$\Pi_\rho := \left\{p \in \Pi; p = \sum_{|k| \leqslant \rho} \alpha_k f_k, \alpha_k \in \mathbb{C}\right\} \tag{2.2}$$

be the subset of polynomials of (radial) degree ρ, where $|k|^2 := \sum_{j=1}^N k_j^2$, $k \in \mathbb{Z}^N$.

For the considerations to follow, the pair H, $\{f_k\}$ is assumed to be fixed. The Banach spaces for which we shall study multipliers of strong convergence are then constructed via the following procedure.

Definition 2.1. Let $\{f_k; k \in \mathbb{Z}^N\}$ be an orthonormal sequence in a Hilbert space H. A Banach space X is called admissible (with respect to the given orthonormal structure H, $\{f_k\}$) if

$$\{f_k\} \subset X, \text{ and } \Pi \text{ is dense in } X, \tag{2.3}$$

$$|(p, f_k)| \leqslant A_k \|p\|_X \qquad (p \in \Pi,\ k \in \mathbb{Z}^N), \tag{2.4}$$

$$\{f_k^*; k \in \mathbb{Z}^N\} \text{ is total on } X. \tag{2.5}$$

Here $f_k^* := f_{k,X}^* \in X^*$ denotes the unique, bounded linear extension of that functional which is generated by f_k via $f_k^*(p) := (p, f_k)$ on $\Pi \subset X$ (cf. (2.3–2.4)). The sequence $\{f_k^*\}$ is then said to be total on X, if $f_k^*(f) = 0$ for all $k \in \mathbb{Z}^N$ and some $f \in X$ necessarily implies $f = 0$.

If Y is a further admissible Banach space, each f_k in the same way generates a functional $f_k^* = \ := f_{k,Y}^* \in Y^*$. By construction, both functionals $f_{k,X}^*, f_{k,Y}^*$ coincide on the dense set Π, thus

$$f_{k,Y}^*(p) = (p, f_k) = f_{k,X}^*(p) \qquad (p \in \Pi)$$

so that we may abbreviate the notation and write f_k^*.

Let s be the set of all sequences $\tau := \{\tau_k; k \in \mathbb{Z}^N\}$ of complex numbers.

DEFINITION 2.2. Let X, Y be admissible. A sequence $\tau := \{\tau_k; k \in \mathbb{Z}^N\} \in s$ is called a multiplier of type (X, Y) if to each $f \in X$ there corresponds $f^\tau \in Y$ such that

$$f_k^*(f^\tau) = \tau_k f_k^*(f) \qquad (k \in \mathbb{Z}^N). \tag{2.6}$$

The set of all multipliers of type (X, Y) is denoted by $M(X, Y)$. Again, $M(X, X) =: M(X)$.

To each $\tau \in M(X, Y)$ one may associate the multiplier operator $T^\tau \in [X, Y]$, given via $T^\tau f := f^\tau$. The set of all multiplier operators is denoted by $[X, Y]_M$. With the natural vector operations and

$$\|\tau\|_{M(X,Y)} := \sup_{\|f\|_X \leq 1} \|f^\tau\|_Y =: \|T^\tau\|_{[X,Y]},$$

$M(X, Y)$ becomes a Banach space, isometrically isomorphic to $[X, Y]_M \subset [X, Y]$. Note that any multiplier of type (X, X) is bounded. For arbitrary $\tau := \{\tau_k; k \in \mathbb{Z}^N\} \in s$ and $\rho > 0$, let the sequences $\tau(\rho) := \{\tau(\rho)_k; k \in \mathbb{Z}^N\}$ be defined by

$$\begin{aligned} \tau(\rho)_k &:= \tau_k, & |k| &\leq \rho, \\ &:= 0, & |k| &> \rho. \end{aligned} \tag{2.7}$$

Then the following assertion is an immediate consequence of the definitions.

LEMMA 2.3. *Let X, Y be admissible. To each $0 \neq \tau \in M(X, Y)$ there exists an index $k_0 \in \mathbb{Z}^N$ and a constant $K = K_\tau > 0$ such that*

$$0 < \|\tau\|_{M(X,Y)} \leq K \, \|\tau(\rho)\|_{M(X,Y)} \tag{2.8}$$

uniformly for $\rho \geq |k_0|$.

For arbitrary subsets $A \subset X$, $B \subset Y$ of admissible Banach spaces X, Y the set $M(A, B)$ of multipliers of type (A, B) is as usual defined via restriction, i.e., $\tau \in M(A, B)$ if to each $f \in A$ there exists $f^\tau \in B$ such that (2.6) holds true. This in particular implies that (settheoretically)

$$M(A_1, B_1) \subset M(A_2, B_2) \tag{2.9}$$

for any $A_2 \subset A_1 \subset X$ and $B_1 \subset B_2 \subset Y$. For further details involving duals of admissible spaces as well as continuous spectral measures one may consult [24, 26] and the literature cited there.

To any element f of an admissible Banach space X one may associate its (unique) Fourier expansion

$$f \sim \sum_{k \in \mathbb{Z}^N} f_k^*(f) f_k. \tag{2.10}$$

The (radial) partial sums of (2.10) are defined by

$$S_\rho f := \sum_{|k| \leqslant \rho} f_k^*(f) f_k \qquad (f \in X,\ \rho \geqslant 0) \tag{2.11}$$

and the Riesz means of order $\alpha \geqslant 0$ by

$$(R, \alpha)_\rho f := \sum_{k \in \mathbb{Z}^N} r_\alpha(|k|/\rho) f_k^*(f) f_k, \tag{2.12}$$

$$\begin{aligned} r_\alpha(t) := [(1-t)_+]^\alpha &:= (1-t)^\alpha, \qquad 0 \leqslant t \leqslant 1, \\ &:= 0, \qquad\qquad t \geqslant 1. \end{aligned}$$

DEFINITION 2.4. An admissible Banach space X is called regular if there exists some $\alpha \geqslant 0$ such that

$$\| r_\alpha(|k|/\rho) \|_{M(X)} \leqslant C_\alpha, \tag{2.13}$$

the constant C_α being independent of $\rho > 0$.

It is important to note that for regular spaces one may derive convenient sufficient criteria for, e.g., radial multipliers. For example, if $\lambda(t)$ is a sufficiently smooth function on $[0, \infty)$ such that $\tau_k = \lambda(|k|)$, such a criterion may be based upon the boundedness of $\int_0^\infty t^\alpha |d\lambda^{(\alpha)}(t)|$. For the details including (X, Y)-criteria, however, we refer to [7, 8, 24, 40] and the literature cited there.

Basic for the present treatment will be the fact that the powerful tool of de La Vallée Poussin (or delayed) means is still available in regular spaces. Indeed, with an arbitrarily often differentiable function λ satisfying

$$\begin{aligned} 0 \leqslant \lambda(t) \leqslant 1, \qquad \lambda(t) &= 1, \qquad 0 \leqslant t \leqslant 1, \\ &= 0, \qquad t \geqslant 2, \end{aligned}$$

(generalized) de La Vallée Poussin means of (2.10) are defined by

$$L_\rho f := \sum_{k \in \mathbb{Z}^N} \lambda(|k|/\rho) f_k^*(f) f_k \qquad (f \in X,\ \rho > 0). \tag{2.14}$$

Let $E_\rho(f; X)$ denote the best approximation of $f \in X$ by polynomials of degree $\rho > 0$, thus

$$E_\rho(f; X) := \inf\{\|f - p\|_X;\ p \in \Pi_\rho\}.$$

LEMMA 2.5. *For an admissible Banach space X let* (2.13) *be satisfied for some* $\alpha \geqslant 0$. *Then the means* (2.14) *possess the properties*

(i) $L_\rho f \in \Pi_{2\rho} \subset H \cap X$ *for each* $f \in X$,

(ii) $L_\rho p = p$ *for each* $p \in \Pi_\rho$,

(iii) $\|L_\rho f\|_X \leqslant D_\alpha \int_0^2 t^{[\alpha]+1} |\lambda^{([\alpha]+2)}(t)| \, dt \, \|f\|_X$, *the constant* D_α *being independent of* $f \in X$, $\rho > 0$,

(iv) $\|L_\rho f - f\|_X \leqslant C E_\rho(f; X)$, *the constant* C *being independent of* $f \in X$, $\rho > 0$.

Proof. Parts (i) and (ii) being obvious, the uniform estimate of (iii) follows by the multiplier criterion mentioned above. Since $\Pi_\rho \subset X$ is a finite dimensional subspace, for any $f \in X$, $\rho > 0$ there certainly exists an element $p_\rho(f) \in \Pi_\rho$ of best approximation, i.e.,

$$E_\rho(f; X) = \|f - p_\rho(f)\|_X, \tag{2.15}$$

so that (iv) follows in view of (ii) and (iii). For further details we refer to [24–26].

Based upon the existence of de La Vallée Poussin means one may establish:

LEMMA 2.6. *Let X, Y be Banach spaces and* $\varepsilon > 0$.

(a) *For any* $T \in [X, Y]$ *there exist* $f = f_{T,\varepsilon} \in X$ *with* $\|f\|_X = 1$ *and* $d \geqslant 0$ *such that*

$$\|T\|_{[X,Y]} = d + \|Tf\|_Y \qquad \text{and} \qquad d < \varepsilon. \tag{2.16}$$

(b) *If X is regular and Y admissible (with respect to the same orthonormal structure H, $\{f_k\}$), then for each* $\tau \in s$ *one has the representation*

$$\|\tau(\rho)\|_{M(X,Y)} = d_\rho + \|T^{\tau(\rho)} w_\rho\|_Y, \tag{2.17}$$

where $d_\rho < \varepsilon$, $w_\rho \in \Pi_{2\rho}$, $\|w_\rho\|_X \leqslant B$ *uniformly for* $\rho > 0$.

Proof. In view of (a), which is obvious, one has for each $\rho > 0$

$$\|\tau(\rho)\|_{M(X,Y)} := \|T^{\tau(\rho)}\|_{[X,Y]} = d_\rho + \|T^{\tau(\rho)} f_\rho\|_Y$$

with $d_\rho < \varepsilon$, $\|f_\rho\|_X = 1$. Now f_ρ may be replaced by some element of $\Pi_{2\rho}$. Indeed, since $T^{\tau(\rho)} f \in \Pi_\rho$ for each $f \in X$ (cf. (2.7)), one has with L_ρ given by (2.14)

$$T^{\tau(\rho)} f_\rho = L_\rho T^{\tau(\rho)} f_\rho = T^{\tau(\rho)} L_\rho f_\rho.$$

Setting $w_\rho := L_\rho f_\rho$, the result follows in view of Lemma 2.5.

3. Sufficient Conditions

Let X, Y be admissible Banach spaces and set (cf. (1.2), (2.11))

$$Y_0 := \{f \in Y; \|S_\rho f - f\|_Y = o(1), \rho \to \infty\}. \tag{3.1}$$

In [24, p. 36] we already generalized those classical results concerned with sequences transforming the whole space into the subset of elements having uniformly convergent Fourier series.

Proposition 3.1. *For admissible X, Y one has* $(\rho \to \infty)$

$$\|\tau(\rho)\|_{M(X,Y)} = \mathcal{O}(1) \Leftrightarrow \tau \in M(X, Y_0).$$

Therefore we are here mainly interested in the case that

$$\sup_{\rho > 0} \|\tau(\rho)\|_{M(X,Y)} = \infty, \tag{3.2}$$

though we do not need to postulate (3.2) explicitly. In this situation the sequence τ can only be expected to transform certain subsets of (sufficiently smooth) elements of X into Y_0. In order to be able to measure smoothness in the present abstract setting, we have to introduce some further notations.

A function $\omega(t)$, defined and continuous on $[0, \infty)$, is called a modulus of continuity if

$$\begin{aligned} &\omega \in C[0, \infty) \text{ monotonically increasing}, \qquad \omega(0) = 0, \\ &\omega(t) > 0 \qquad \text{for } t > 0, \qquad \omega(t_1 + t_2) \leqslant \omega(t_1) + \omega(t_2). \end{aligned} \tag{3.3}$$

It has the familiar properties (cf. [37, p. 96 ff])

$$\begin{aligned} &\text{(i)} \quad \omega(st) \leqslant (1 + s)\,\omega(t) \qquad \text{for any } s, t \geqslant 0, \\ &\text{(ii)} \quad \omega(t_2)/t_2 \leqslant 2\omega(t_1)/t_1 \qquad \text{for any } t_2 \geqslant t_1 > 0. \end{aligned} \tag{3.4}$$

As a measure for the rate of convergence we often use functions φ satisfying

$$\begin{aligned} &\varphi \in C(0, \infty] \text{ monotonically decreasing}, \qquad \varphi(\infty) = 0, \\ &\varphi(\rho) > 0 \qquad \text{for } \rho > 0, \qquad \varphi(\rho) \leqslant D\varphi(2\rho), \end{aligned} \tag{3.5}$$

the constant D being independent of $\rho > 0$. Let us set

$$E_\varphi := \{f \in X; E_\rho(f; X) = \mathcal{O}(\varphi(\rho)), \rho \to \infty\}. \tag{3.6}$$

The order of best approximation is a standard tool to measure smoothness, even in the classical one-dimensional trigonometric situation. A more direct way, however, is provided by the K-functional, defined for $f \in X$, $t \geqslant 0$ by

$$K(X, Z; f, t) := \inf_{g \in Z} (\|f - g\|_X + t\,|g|_Z), \tag{3.7}$$

where $Z \subset X$ is a subspace with seminorm $|\cdot|_Z$. In many cases of interest it is equivalent to some modulus of continuity, see Section 6. The K-functional defines a seminorm on X for each $t \geqslant 0$ and satisfies the elementary properties

$$\begin{aligned} K(X, Z; f, t) &\leqslant \|f\|_X \qquad \text{for any } f \in X, \\ &\leqslant t\,|f|_Z \qquad \text{for any } f \in Z. \end{aligned} \tag{3.8}$$

For a modulus of continuity ω we set

$$X_\omega := \{f \in X;\ K(X, Z; f, t) = \mathcal{O}(\omega(t)),\ t \to 0+\}. \tag{3.9}$$

THEOREM 3.2. *Let X be regular and Y admissible (with respect to the same orthonormal structure H, $\{f_k\}$). If ω, φ satisfy* (3.3), (3.5), *respectively, then*

$$\begin{aligned} &\text{(i)} \quad \tau \in M(X, Y), \\ &\text{(ii)} \quad \omega(\varphi(\rho))\,\|\tau(\rho)\|_{M(X,Y)} = o(1) \qquad (\rho \to \infty) \end{aligned} \tag{3.10}$$

implies $\tau \in M(E_{\omega(\varphi)}, Y_0)$.

Proof. The assertion being trivial for $\tau = 0$, let $0 \neq \tau \in M(X, Y)$. Then for $f \in E_{\omega(\varphi)}$ the assumptions imply by Lemma 2.3 and (2.15) that for $\rho \to \infty$

$$\begin{aligned} \|S_\rho f^\tau - f^\tau\|_Y &\leqslant \|S_\rho T^\tau f - S_\rho T^\tau p_\rho(f)\|_Y + \|T^\tau p_\rho(f) - T^\tau f\|_Y \\ &\leqslant (\|\tau(\rho)\|_{M(X,Y)} + \|\tau\|_{M(X,Y)})\, E_\rho(f; X) \\ &\leqslant (1 + K)\,\|\tau(\rho)\|_{M(X,Y)}\, \omega(\varphi(\rho)) = o(1). \end{aligned}$$

This completes the proof.

COROLLARY 3.3. *Under the assumptions of Theorem* 3.2 *one has*

$$\|S_\rho f^\tau - f^\tau\|_Y \leqslant A\omega(\varphi(\rho))\,\|\tau(\rho)\|_{M(X,Y)}.$$

This signifies even an order of approximation for the partial sums.

Structural properties of the element $f \in X$ as measured in terms of (3.6) or (3.9) may indeed be connected via an inequality of Jackson-type ($A \geqslant 1$)

$$E_\rho(g; X) \leqslant A\varphi(\rho)\,|g|_Z \qquad (g \in Z). \tag{3.11}$$

In fact, for any $f \in X$ and $g \in Z \subset X$ one then has

$$E_\rho(f; X) \leqslant E_\rho(f-g; X) + E_\rho(g; X) \leqslant \|f-g\|_X + A\varphi(\rho)\,|g|_Z,$$

and therefore in view of (3.7)

$$E_\rho(f; X) \leqslant AK(X, Z; f, \varphi(\rho)). \tag{3.12}$$

Hence $X_\omega \subset E_{\omega(\varphi)}$, and an application of Theorem 3.2 delivers

COROLLARY 3.4. *If the Jackson-type inequality* (3.11) *holds true, then the assumptions of Theorem* 3.2 *also imply* $\tau \in M(X_\omega, Y_0)$.

As mentioned in the Introduction, the present approach admits a parallel treatment of multipliers of uniform boundedness. To formulate the results, let (cf. (3.1), (3.6), (3.9))

$$Y_b := \{f \in Y; \|S_\rho f\|_Y = \mathcal{O}(1), \rho \to \infty\}, \tag{3.13}$$

$$E_{\varphi,0} := \{f \in X; E_\rho(f; X) = o(\varphi(\rho)), \rho \to \infty\}, \tag{3.14}$$

$$X_{\omega,0} := \{f \in X; K(X, Z; f, t) = o(\omega(t)), t \to 0+\}. \tag{3.15}$$

THEOREM 3.5. *Let X be regular and Y admissible. If* ω, φ *satisfy* (3.3), (3.5), *respectively, then*

$$\begin{aligned} &\text{(i)} \quad \tau \in M(X, Y),\\ &\text{(ii)} \quad \omega(\varphi(\rho))\,\|\tau(\rho)\|_{M(X,Y)} = \mathcal{O}(1) \qquad (\rho \to \infty) \end{aligned} \tag{3.16}$$

implies $\tau \in M(E_{\omega(\varphi),0}, Y_0) \subset M(E_{\omega(\varphi),0}, Y_b)$, *as well as* $\tau \in M(E_{\omega(\varphi)}, Y_b)$.

COROLLARY 3.6. *If the Jackson-type inequality* (3.11) *holds true, then the assumptions of Theorem* 3.5 *also imply* $\tau \in M(X_{\omega,0}, Y_0) \subset M(X_{\omega,0}, Y_b)$, *as well as* $\tau \in M(X_\omega, Y_b)$.

Let us observe that in this section, thus for sufficiency, we actually did not use the growth condition $\varphi(\rho) \leqslant D\varphi(2\rho)$ (cf. (3.5)).

4. NECESSARY CONDITIONS

Let us first consider the case that

$$\omega(\varphi(\rho)) = \mathcal{O}(\varphi(\rho)) \qquad (\rho \to \infty), \tag{4.1}$$

where ω, φ satisfy (3.3), (3.5), respectively. Here we only mention a rather simple result.

PROPOSITION 4.1. *Let X, Y be admissible. If* (4.1) *holds true and*

$$\|S_\rho\|_{[X]} = o(1/\varphi(\rho)) \qquad (\rho \to \infty), \tag{4.2}$$

then $\tau \in M(X, Y)$ always satisfies condition (3.10ii).

Indeed, the assumptions imply $(\rho \to \infty)$

$$\begin{aligned}\omega(\varphi(\rho)) \, \|\tau(\rho)\|_{M(X,Y)} &\leqslant A_1 \varphi(\rho) \, \|T^\tau S_\rho\|_{[X,Y]} \\ &\leqslant A_1 \, \|\tau\|_{M(X,Y)} \, \|S_\rho\|_{[X]} \, \varphi(\rho) = o(1).\end{aligned}$$

Though this covers the relevant situation for the one-dimensional trigonometric system (see Section 6.1), condition (4.2) is rather striking, also from the point of view of further applications (cf. Sections 6.2–6.3). Therefore the following considerations practically exclude those moduli of continuity satisfying the (saturation) condition $\omega(t) = \mathcal{O}(t)$ (see, however, Remark 4.6, Corollary 4.9).

In fact, the present investigations are mainly concerned with the case that

$$\sup_{\rho > 0} \omega(\varphi(\rho))/\varphi(\rho) = \infty, \tag{4.3}$$

again for ω, φ subject to (3.3), (3.5), respectively. Here we assume a Bernstein inequality of the type

$$|p_\rho|_Z \leqslant (1/\varphi(\rho)) \, \|p_\rho\|_X \qquad (p_\rho \in \Pi_\rho), \tag{4.4}$$

where Z with seminorm $|\cdot|_Z$ is a subspace of an admissible X satisfying $\Pi \subset Z \subset X$, and φ is any function subject to (3.5). The necessity is then shown via the construction of some counterexample. To this end, let us first establish

PROPOSITION 4.2. *Let X be regular, and ω, φ be such that* (4.3) *holds true. Suppose that the Bernstein-type inequality* (4.4) *is satisfied. If $\{\rho_k; k \in \mathbb{N}_0\}$ is a sequence of positive numbers monotonically increasing to infinity such that*

$$\sum_{i=0}^{k-1} \omega(\varphi(\rho_i))/\varphi(\rho_i) \leqslant \omega(\varphi(\rho_k))/\varphi(\rho_k), \tag{4.5}$$

$$\omega(\varphi(\rho_k)) \leqslant (1/2) \, \omega(\varphi(\rho_{k-1})), \tag{4.6}$$

then the element

$$w := \sum_{k=0}^{\infty} h_k, \qquad h_k := \omega(\varphi(\rho_k)) \, w_{\rho_k}, \tag{4.7}$$

belongs to X_ω for any $w_\rho \in \Pi_{2\rho}$ with $\|w_\rho\|_X \leqslant B$.

Proof. Obviously, (4.5)–(4.6) are meaningful in view of (4.3). By (4.6)

$$\|w\|_X \leqslant \sum_{k=0}^{\infty} \|h_k\|_X \leqslant B \sum_{k=0}^{\infty} \omega(\varphi(\rho_k)) \leqslant B\omega(\varphi(\rho_0)) \sum_{k=0}^{\infty} 2^{-k} < \infty,$$

so that w as given by (4.7) is well defined as an element of X. Moreover, it is sufficient to consider $t \in (0, a)$, where $a > 0$ is such that (cf. (3.3), (3.5))

$$(0, a) \subset (0, \varphi(\rho_0)) \cap \{t > 0; \omega(t) < 1\}. \tag{4.8}$$

Then for each $t \in (0, a)$ there exists $k \in \mathbb{N}_0$ such that

$$\varphi(\rho_{k+1}) \leqslant t < \varphi(\rho_k). \tag{4.9}$$

It follows that

$$K(X, Z; w, t) \leqslant \sum_{i=0}^{k} K(X, Z; h_i, t) + \sum_{i=k+1}^{\infty} K(X, Z; h_i, t).$$

Since $w_\rho \in \Pi_{2\rho} \subset Z$, one has by (3.5), (3.8), (4.4) that for $0 \leqslant i \leqslant k$

$$\begin{aligned} K(X, Z; h_i, t) &= \omega(\varphi(\rho_i))\, K(X, Z; w_{\rho_i}, t) \\ &\leqslant \omega(\varphi(\rho_i))\, t\, |w_{\rho_i}|_Z \leqslant \omega(\varphi(\rho_i))\, t\, \|w_{\rho_i}\|_X/\varphi(2\rho_i) \\ &\leqslant BDt\omega(\varphi(\rho_i))/\varphi(\rho_i), \end{aligned} \tag{4.10}$$

whereas for $i \geqslant k+1$

$$K(X, Z; h_i, t) \leqslant \omega(\varphi(\rho_i))\, \|w_{\rho_i}\|_X \leqslant B\omega(\varphi(\rho_i)). \tag{4.11}$$

Therefore by (4.5–4.6)

$$\begin{aligned} K(X, Z; w, t) &\leqslant BDt\omega(\varphi(\rho_k))/\varphi(\rho_k) \\ &\quad + BDt \sum_{i=0}^{k-1} \omega(\varphi(\rho_i))/\varphi(\rho_i) + B \sum_{i=k+1}^{\infty} \omega(\varphi(\rho_i)) \\ &\leqslant 2BDt\omega(\varphi(\rho_k))/\varphi(\rho_k) + B\omega(\varphi(\rho_{k+1})) \sum_{i=k+1}^{\infty} 2^{k+1-i}. \end{aligned}$$

Since in view of (3.4ii), (4.9)

$$\begin{aligned} \omega(\varphi(\rho_k))/\varphi(\rho_k) &\leqslant 2\omega(t)/t, \\ \omega(\varphi(\rho_{k+1})) &\leqslant \omega(t), \end{aligned} \tag{4.12}$$

one finally obtains the estimate

$$K(X, Z; w, t) \leqslant 4BD\omega(t) + 2B\omega(t) = \mathcal{O}(\omega(t)),$$

thus $w \in X_\omega$ (cf. (3.9)).

THEOREM 4.3. *Let X be regular and Y admissible (with respect to the same orthonormal structure H, $\{f_k\}$). Let ω, φ satisfy* (4.3), *and suppose that the Bernstein-type inequality* (4.4) *holds true. Then $\tau \in M(X_\omega, Y_0)$ necessarily implies* (3.10ii).

Proof. Suppose that (3.10ii) does not hold. Then there exist $\varepsilon_0 > 0$ and some sequence $\{\rho_{0m}; m \in \mathbb{N}_0\}$, monotonically increasing to infinity, such that $\omega(\varphi(\rho_{0m})) \leqslant 1$ and

$$\omega(\varphi(\rho_{0m})) \,\|\tau(\rho_{0m})\|_{M(X,Y)} \geqslant \varepsilon_0 \qquad (m \in \mathbb{N}_0). \tag{4.13}$$

Now the assertion follows if we can construct $w \in X_\omega$ such that for some subsequence $\{\rho_k; k \in \mathbb{N}_0\} \subset \{\rho_{0m}; m \in \mathbb{N}_0\}$ with $2\rho_{k-1} < \rho_k$ one has $(k \to \infty)$

$$\|S_{\rho_k} w^\tau - S_{2\rho_{k-1}} w^\tau\|_Y \geqslant \omega(\varphi(\rho_k)) \,\|\tau(\rho_k)\|_{M(X,Y)} + o(1); \tag{4.14}$$

for then $w^\tau \in Y_0$ would be impossible.

In order to show (4.14) we commence with the representation (2.17) for $\varepsilon := \varepsilon_0/2$. In connection with the family $\{w_\rho\}$ used there we now select a subsequence $\{\rho_k\} \subset \{\rho_{0m}\}$ via the following procedure: Let $\rho_0 := \rho_{00}$, and suppose that $\rho_0, ..., \rho_{k-1}$ have already been determined. Then $\rho_k \in \{\rho_{0m}\}$ is chosen such that $(k \neq 0)$

$$\begin{aligned}
&\text{(i)} \quad 2\rho_{k-1} < \rho_k,\\
&\text{(ii)} \quad \sum_{i=0}^{k-1} \omega(\varphi(\rho_i))/\varphi(\rho_i) \leqslant \omega(\varphi(\rho_k))/\varphi(\rho_k),\\
&\text{(iii)} \quad \omega(\varphi(\rho_k)) \leqslant \omega(\varphi(\rho_{k-1}))/2.\\
&\text{(iv)} \quad \omega(\varphi(\rho_k))\{\|\tau(\rho_{k-1})\|_{M(X,Y)} + \|\tau(2\rho_{k-1})\|_{M(X,Y)}\} \leqslant 1/k.
\end{aligned} \tag{4.15}$$

Then by Lemma 2.6, Proposition 4.2

$$w := \sum_{k=0}^{\infty} h_k, \qquad h_k := \omega(\varphi(\rho_k))\, w_{\rho_k}$$

is well defined as an element of X_ω, and thus $w^\tau \in Y_0 \subset Y$ by assumption. It follows that

$$\begin{aligned}
S_{\rho_k} w^\tau - S_{2\rho_{k-1}} w^\tau &= \sum_{|j| \leqslant \rho_k} \tau_j f_j^*(h_k) f_j\\
&\quad + \sum_{|j| \leqslant \rho_k} \tau_j f_j^* \Big(\sum_{i \neq k} h_i\Big) f_j - \sum_{|j| \leqslant 2\rho_{k-1}} \tau_j f_j^* \Big(\sum_{i=0}^{\infty} h_i\Big) f_j\\
&= \sum_{|j| \leqslant \rho_k} \tau_j f_j^*(h_k) f_j + \sum_{|j| \leqslant \rho_k} \tau_j f_j^* \Big(\sum_{i=k+1}^{\infty} h_i\Big) f_j\\
&\quad - \sum_{|j| \leqslant 2\rho_{k-1}} \tau_j f_j^* \Big(\sum_{i=k}^{\infty} h_i\Big) f_j =: I_1 + I_2 + I_3,
\end{aligned} \tag{4.16}$$

say, where we have used $\sum_{i=0}^{k-1} h_i \in \Pi_{2\rho_{k-1}}$ and (4.15i). In view of (2.17) with $\varepsilon := \varepsilon_0/2$ and $\omega(\varphi(\rho_k)) \leqslant 1$ one has (cf. (2.7))

$$\begin{aligned}\|I_1\|_Y &= \omega(\varphi(\rho_k)) \| \sum_{|j| \leqslant \rho_k} \tau_j f_j^*(w_{\rho_k}) f_j \|_Y \\ &= \omega(\varphi(\rho_k)) \, \| T^{\tau(\rho_k)} w_{\rho_k} \|_Y \\ &= \omega(\varphi(\rho_k)) [\| T^{\tau(\rho_k)} \|_{[X,Y]} - d_{\rho_k}] \\ &\geqslant \omega(\varphi(\rho_k)) \, \| \tau(\rho_k) \|_{M(X,Y)} - \omega(\varphi(\rho_k)) \varepsilon_0/2 \geqslant \varepsilon_0/2, \end{aligned} \tag{4.17}$$

$$\begin{aligned}\|I_2 + I_3\|_Y = \Bigg\| & \sum_{i=k+1}^{\infty} \omega(\varphi(\rho_i)) \sum_{|j| \leqslant \rho_k} \tau_j f_j^*(w_{\rho_i}) f_j \\ & - \sum_{i=k}^{\infty} \omega(\varphi(\rho_i)) \sum_{|j| \leqslant 2\rho_{k-1}} \tau_j f_j^*(w_{\rho_i}) f_j \Bigg\|_Y \\ \leqslant & \sum_{i=k+1}^{\infty} \omega(\varphi(\rho_i)) \, \| T^{\tau(\rho_k)} w_{\rho_i} \|_Y + \sum_{i=k}^{\infty} \omega(\varphi(\rho_i)) \, \| T^{\tau(2\rho_{k-1})} w_{\rho_i} \|_Y \\ \leqslant & B \, \| \tau(\rho_k) \|_{M(X,Y)} \sum_{i=k+1}^{\infty} \omega(\varphi(\rho_i)) + B \, \| \tau(2\rho_{k-1}) \|_{M(X,Y)} \sum_{i=k}^{\infty} \omega(\varphi(\rho_i)) \\ \leqslant & 2B \, \| \tau(\rho_k) \|_{M(X,Y)} \omega(\varphi(\rho_{k+1})) + 2B \, \| \tau(2\rho_{k-1}) \|_{M(X,Y)} \omega(\varphi(\rho_k)) \\ \leqslant & 2B/(k+1) + 2B/k \leqslant 4B/k. \end{aligned} \tag{4.18}$$

This establishes (4.14), and hence the assertion of the theorem.

Let us observe that the argument (cf. (4.16)) depends heavily upon the fact that

$$T^{\tau(\rho_k)} \left(\sum_{i=0}^{k-1} h_i \right) - T^{\tau(2\rho_{k-1})} \left(\sum_{i=0}^{k-1} h_i \right) = 0. \tag{4.18*}$$

This may be compared with the treatment of the standard uniform boundedness principle via the gliding hump method (cf. [6, p. 18]).

COROLLARY 4.4. *Let X be regular and Y admissible. Suppose that the Jackson-type inequality* (3.11) *and the Bernstein-type inequality* (4.4) *hold true for some $Z \subset X$ and for ω, φ subject to* (4.3). *Then $\tau \in M(E_{\omega(\varphi)}, Y_0)$ necessarily implies* (3.10ii).

In fact, the Jackson-type inequality now in addition guarantees $X_\omega \subset E_{\omega(\varphi)}$ (cf. (3.12)).

For a parallel treatment of the necessity of (3.16ii) we need

PROPOSITION 4.5. *Let X be regular and ω, φ be such that* (4.3) *holds true. Suppose that the Bernstein-type inequality* (4.4) *is satisfied. If*

$\{\rho_k; k \in \mathbb{N}\}$ is a sequence of positive numbers monotonically increasing to infinity such that

$$\sum_{i=1}^{k-1} \omega(\varphi(\rho_i))/i^2\varphi(\rho_i) \leqslant \omega(\varphi(\rho_k))/k^2\varphi(\rho_k), \tag{4.19}$$

$$\omega(\varphi(\rho_k)) \leqslant \omega(\varphi(\rho_{k-1}))/2, \tag{4.20}$$

then the element

$$w := \sum_{k=1}^{\infty} h_k, \qquad h_k := k^{-2}\omega(\varphi(\rho_k))\, w_{\rho_k} \tag{4.21}$$

belongs to $X_{\omega,0}$ for any $w_\rho \in \Pi_{2\rho}$ with $\|w_\rho\|_X \leqslant B$.

Proof. Proceeding as in the proof of Proposition 4.2, the counterparts to (4.10)–(4.11) now read

$$\begin{aligned} K(X, Z; h_i, t) &\leqslant Bi^{-2}\omega(\varphi(\rho_i))\, Dt/\varphi(\rho_i), \qquad && 1 \leqslant i \leqslant k, \\ &\leqslant Bi^{-2}\omega(\varphi(\rho_i)) && i \geqslant k+1, \end{aligned}$$

so that one has by (4.12)

$$\begin{aligned} K(X, Z; w, t) &\leqslant BDt\omega(\varphi(\rho_k))/k^2\varphi(\rho_k) \\ &\quad + BDt \sum_{i=1}^{k-1} \omega(\varphi(\rho_i))/i^2\varphi(\rho_i) + B \sum_{i=k+1}^{\infty} \omega(\varphi(\rho_i))/i^2 \\ &\leqslant 2BDt\omega(\varphi(\rho_k))/k^2\varphi(\rho_k) + B\omega(\varphi(\rho_{k+1})) \sum_{i=k+1}^{\infty} 2^{k+1-i}/i^2 \\ &\leqslant 2Bk^{-2}[Dt\omega(\varphi(\rho_k))/\varphi(\rho_k) + \omega(\varphi(\rho_{k+1}))] \\ &\leqslant 2Bk^{-2}[2D+1]\,\omega(t). \end{aligned}$$

This implies $K(X, Z; w, t) = o(\omega(t))$ (cf. (4.9)), thus $w \in X_{\omega,0}$.

Remark 4.6. Note that in connection with $X_{\omega,0}$ condition (4.3) is no essential restriction, for otherwise one has $X_{\omega,0} = \{f \in X;\ K(X, Z; f, t) = o(t)\}$, a trivial class.

THEOREM 4.7. *Let X be regular and Y admissible. Let ω, φ satisfy* (4.3), *and suppose that the Bernstein-type inequality* (4.4) *holds true. Then $\tau \in M(X_{\omega,0}, Y_0)$ or $\tau \in M(X_{\omega,0}, Y_b)$ or $\tau \in M(X_\omega, Y_b)$ necessarily implies* (3.16ii).

Proof. We proceed as in the proof of Theorem 4.3. If (3.16ii) does not hold, then to each bound k^3 there exists a sequence $\{\rho_{k,m}; m \in \mathbb{N}\}$, monotonically increasing to infinity, such that $\omega(\varphi(\rho_{k,m})) \leqslant 1$ and

$$\omega(\varphi(\rho_{k,m})) \, \|\tau(\rho_{k,m})\|_{M(X,Y)} \geqslant k^3 \qquad (k, m \in \mathbb{N}). \tag{4.22}$$

Again the proof follows by constructing some $w \in X_{\omega,0}$ such that for a subsequence $\{\rho_k; k \in \mathbb{N}\}$, monotonically increasing to infinity with $\rho_k \in \{\rho_{k,m}\}$ and $2\rho_{k-1} < \rho_k$, one has $(k \to \infty)$

$$\|S_{\rho_k} w^\tau - S_{2\rho_{k-1}} w^\tau\|_Y \geqslant k^{-2}\omega(\varphi(\rho_k)) \, \|\tau(\rho_k)\|_{M(X,Y)} + o(1); \tag{4.23}$$

for then neither $w^\tau \in Y_0$ nor $w^\tau \in Y_b$ would be possible.

In order to verify (4.23) we again start off with Lemma 2.6(b) for $\varepsilon = 1/2$ and select a subsequence $\{\rho_k\}$, monotonically increasing to infinity, according to the following procedure: Let $\rho_1 := \rho_{1,1}$, and suppose that $\rho_1, ..., \rho_{k-1}$ have already been determined. Then choose $\rho_k \in \{\rho_{k,m}\}$ such that (4.15i, iii, iv) hold true, whereas (ii) is replaced by (4.19). Hence w as given by (4.21) belongs to $X_{\omega,0} \subset X_\omega$. Proceeding via (4.16) we now have (cf. (4.17))

$$\begin{aligned}\|I_1\|_Y := k^{-2}\omega(\varphi(\rho_k)) \left\| \sum_{|j| \leqslant \rho_k} \tau_j f_j^*(w_{\rho_k}) f_j \right\|_Y \\ \geqslant k^{-2}\omega(\varphi(\rho_k)) \, \|\tau(\rho_k)\|_{M(X,Y)} - \omega(\varphi(\rho_k))/2k^2,\end{aligned}$$

as well as (4.18); this establishes (4.23), and hence the theorem.

Corollary 4.8. *Let X be regular and Y admissible. Suppose that the Jackson-type inequality* (3.11) *and the Bernstein-type inequality* (4.4) *hold true for some $Z \subset X$ and for ω, φ subject to* (4.3). *Then $\tau \in M(E_{\omega(\varphi),0}, Y_0)$ or $\tau \in M(E_{\omega(\varphi),0}, Y_b)$ or $\tau \in M(E_{\omega(\varphi)}, Y_b)$ necessarily implies* (3.16ii).

Let us conclude with the observation that in connection with the "$\mathcal{O}$"-condition (3.16ii) we actually do not need (4.3) (cf. Remark 4.6). For example, if one is interested in $M(X_\omega, Y_b)$ for a modulus of continuity satisfying $\omega(t) = \mathcal{O}(t)$, then (4.19) in fact reduces to

$$\sum_{i=1}^{k-1} \omega(\varphi(\rho_i))/i^2\varphi(\rho_i) \leqslant A \sum_{i=1}^{k-1} i^{-2} = \mathcal{O}(1) \qquad (k \to \infty),$$

and the proof of Proposition 4.5 shows that w defined by (4.21) belongs to X_ω even if $\omega(t) = \mathcal{O}(t)$ (cf. (3.4ii)). Thus, without modifying the proof of Theorem 4.7, one also obtains

Corollary 4.9. *Let X be regular, Y admissible, and suppose that* (4.4) *holds true. Then for any* ω *subject to* (3.3) *the condition* $\tau \in M(X_\omega, Y_b)$ *necessarily implies* (3.16ii).

As already indicated by these considerations, it is the "o"-case of (3.10ii), thus Theorem 4.3, Corollary 4.4, which is the most interesting one in connection with multipliers of strong convergence. In fact, based upon Lemma 2.6, the necessity of the "$\mathcal{O}$"-condition (3.16ii) may also be derived via a suitable application of the classical uniform boundedness principle. In this note, however, we prefered the parallel treatment via Proposition 4.5.

5. An Equivalence Theorem

Let us collect the results of the previous sections according to the following equivalence assertion.

THEOREM 5.1. *Let X be regular and Y admissible* (*with respect to the same orthonormal structure H*, $\{f_k\}$). *Suppose that the Jackson-type inequality* (3.11) *as well as the Bernstein-type inequality* (4.4) *hold true for some* $Z \subset X$ *and for* ω, φ *subject to* (4.3) *or* (4.1–4.2). *If* $\tau \in M(X, Y)$, *then* $(\rho \to \infty)$:

$$\text{(a)}\quad \omega(\varphi(\rho))\, \|\tau(\rho)\|_{M(X,Y)} = o(1) \Leftrightarrow \tau \in M(E_{\omega(\varphi)}, Y_0) \Leftrightarrow \tau \in M(X_\omega, Y_0).$$

$$\text{(b)}\quad \omega(\varphi(\rho))\, \|\tau(\rho)\|_{M(X,Y)} = \mathcal{O}(1) \Leftrightarrow \tau \in M(E_{\omega(\varphi),0}, Y_0) \Leftrightarrow \tau \in M(X_{\omega,0}, Y_0) \Leftrightarrow \tau \in M(E_{\omega(\varphi),0}, Y_b) \Leftrightarrow \tau \in M(X_{\omega,0}, Y_b) \Leftrightarrow \tau \in M(E_{\omega(\varphi)}, Y_b) \Leftrightarrow \tau \in M(X_\omega, Y_b).$$

In connection with the "o"-case of part (a) we mention that one part, namely sufficiency, may indeed be formulated so as to yield convergence assertions with rates (cf. Corollary 3.3). On the other hand, the present procedure via (4.14) to prove necessity then does not seem to be suitable enough to regain the correct order of approximation. We also emphasize that the results of Section 4 do not use the a priori assumption $\tau \in M(X, Y)$, in contrast to Section 3. Moreover, since we are mainly interested in counterparts of the classical equivalence Theorem 1.1, thus in Theorem 5.1(a), we have assumed that the subspace $Z \subset X$ as well as the order φ are the same in the basic inequalities (3.11), (4.4). But this is by no means necessary as is indicated by the separate treatment of the topics of Sections 3 and 4.

Concerning the "$\mathcal{O}$"-case of part (b), we recall Remark 4.6, Corollary 4.9 and formulate

COROLLARY 5.2. *Let X be regular, Y admissible, and suppose that the inequalities* (3.11), (4.4) *hold true for some $Z \subset X$ and ω, φ subject to* (3.3), (3.5), *respectively. If $\tau \in M(X, Y)$, then*

$$\omega(\varphi(\rho)) \, \|\tau(\rho)\|_{M(X,Y)} = \mathcal{O}(1) \Leftrightarrow \tau \in M(X_\omega, Y_b).$$

6. Applications

It is almost obvious that the abstract equivalence Theorem 5.1 admits immediate applications in connection with various expansions. Here we only wish to indicate some of them. First, we discuss how to recapture the classical results. Then we give some new applications to the multivariate trigonometric system and to Jacobi expansions. In any case, emphasis is laid upon the verification of the assumptions posed in Theorem 5.1 rather than upon an explicit formulation of results (for a detailed treatment one may consult [20]).

6.1. *A Review of Classical Results*

Let $X_{2\pi}$ be one of the spaces $C_{2\pi}$ or $L^p_{2\pi}$, $1 \leqslant p < \infty$, where

$$\|f\|_{p,2\pi} := \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(u)|^p \, du\right)^{1/p}, \qquad 1 \leqslant p < \infty,$$
$$:= \operatorname*{ess\,sup}_{u \in \mathbb{R}} |f(u)|, \qquad p = \infty.$$

With respect to the orthonormal structure $H := L^2_{2\pi}$, $\{f_k\} := \{e^{iku};\ k \in \mathbb{Z}\}$, the spaces $X_{2\pi}$ are admissible and even regular, e.g., with $\alpha = 1$ by Fejér's theorem. Of course, the corresponding functional f_k^* is given by the kth Fourier coefficient $f^\wedge(k)$ of f (cf. (1.1)). Let

$$Z_{2\pi} = X^1_{2\pi} := \{g \in X_{2\pi};\ g' \in X_{2\pi}\}$$

be the Sobolev space (of order 1) with $|g|_{Z_{2\pi}} := \|g'\|_{X_{2\pi}}$. Then one has the Jackson inequality

$$E_\rho(g; X_{2\pi}) \leqslant (A/\rho) \, \|g'\|_{X_{2\pi}} \qquad (g \in X^1_{2\pi}) \tag{6.1}$$

and the Bernstein inequality

$$\|p'_\rho\|_{X_{2\pi}} \leqslant \rho \, \|p_\rho\|_{X_{2\pi}} \qquad (p_\rho \in \Pi_\rho),$$

thus $\varphi(\rho)=\rho^{-1}$ (cf. [6, pp. 97, 99]). Since $K(X_{2\pi}, Z_{2\pi}; f, t)$ is equivalent to the first order modulus of continuity (cf. [4, p. 192 ff]), the spaces $(X_{2\pi})_\omega$ may be characterized via (cf. (1.3)–(1.4))

$$(X_{2\pi})_\omega := \{f \in X_{2\pi};\ \sup_{|h| \leqslant t} \|f(u+h) - f(u)\|_{X_{2\pi}} = \mathcal{O}(\omega(t))\}.$$

Classical results ensure the sharp estimates

$$\|S_\rho\|_{[X_{2\pi}]} = \begin{cases} \mathcal{O}(\log \rho) & \text{for } X_{2\pi} = C_{2\pi}, L^1_{2\pi}, \\ \mathcal{O}(1) & \text{for } X_{2\pi} = L^p_{2\pi},\ 1 < p < \infty, \end{cases} \tag{6.2}$$

so that in view of (6.1) condition (4.2) is always satisfied. Therefore an application of Theorem 5.1(a) to $X = Y = C_{2\pi}$ immediately regains Theorem 1.1 since $\|\tau(n)\|_{M(C_{2\pi}, C_{2\pi})} = \|D_n^\tau\|_1$ (cf. [6, p. 266 ff]). Let us observe that in the classical setting instead of the a priori assumption $\tau \in M(C_{2\pi}, C_{2\pi})$ one uses the equivalent one (cf. [6, p. 232 ff, 266 ff]) that $\|\sum_{k=0}^n D_k^\tau\|_1 = \mathcal{O}(n)$ or that τ is a Fourier–Stieltjes sequence. At the same time Theorem 5.1 recovers results of DeVore [10], for example,

COROLLARY 6.1. *For $\tau \in M(C_{2\pi}, C_{2\pi})$ one has*

$$\omega(1/n)\|D_n^\tau\|_1 = \mathcal{O}(1) \Leftrightarrow \tau \in M((C_{2\pi})_{\omega,0}, (C_{2\pi})_0).$$

Another choice of admissible spaces is provided by $X = L^1_{2\pi}$, $Y = C_{2\pi}$. Then, using (cf. [6, p. 273])

$$M(L^p_{2\pi}, C_{2\pi}) = (L^{p'}_{2\pi})^\wedge := \{\sigma \in s; \sigma_k = f^\wedge(k), f \in L^{p'}_{2\pi}\},$$

$(1/p) + (1/p') = 1$, Theorem 5.1 delivers results of Pochuev [29], for example,

COROLLARY 6.2. *If $\tau \in (L^\infty_{2\pi})^\wedge$, then*

$$\omega(1/n)\|D_n^\tau\|_{C_{2\pi}} = \mathcal{O}(1) \Leftrightarrow \tau \in M((L^1_{2\pi})_\omega, (C_{2\pi})_0).$$

6.2. *Multivariate Trigonometric System*

Now let $X_{2\pi}$ be one of the corresponding spaces of functions, 2π-periodic in each of the N real variables, where, e.g.,

$$\|f\|_{p,2\pi} := \left(\frac{1}{(2\pi)^N}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}|f(u)|^p\,du\right)^{1/p} \qquad (1 \leqslant p < \infty).$$

Again $Z_{2\pi} := X^r_{2\pi}$ denotes the Sobolev space of functions with all (distributional) partial derivatives up to the order $r \in \mathbb{N}_0$ belonging to $X_{2\pi}$ with $|g|_{Z_{2\pi}} := \sum_{|m|=r}\|D^m g\|_{X_{2\pi}}$. Obviously, $X_{2\pi}$ is admissible with respect to $L^2_{2\pi}$, $\{e^{iku}; k \in \mathbb{Z}^N\}$, and it follows by classical results of Bochner and Stein (cf. [33, p. 255 ff]) that the N-dimensional trigonometric system is Riesz bounded (cf. (2.13)) in $L^p_{2\pi}$ if $\alpha > (N-1)\,|(1/p)-(1/2)|$. Moreover, in view of [16, 30]

$$\|S_\rho\|_{[L^p_{2\pi}]} \leqslant \left\| \sum_{|k|\leqslant \rho} e^{iku} \right\|_1 = \mathcal{O}(\rho^{(N-1)/2}), \tag{6.3}$$

the latter estimate being sharp for $p=1$ (and $p=\infty$). One has the Jackson inequality (cf. [5, 27, p. 186 ff; 40, p. 79])

$$E_\rho(g; X_{2\pi}) \leqslant A\rho^{-r}\,|g|_{X^r_{2\pi}} \qquad (g \in X^r_{2\pi}) \tag{6.4}$$

and the Bernstein inequality

$$|p_\rho|_{X^r_{2\pi}} \leqslant C\rho^r\,\|p_\rho\|_{X_{2\pi}} \qquad (p_\rho \in \Pi_\rho).$$

Let the rth (radial) modulus of continuity of $f \in X_{2\pi}$ be given by

$$\omega_r(X_{2\pi}; f; t) := \sup_{|h|\leqslant t} \left\| \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} f(u+jh) \right\|_{X_{2\pi}},$$

and let $\mathrm{Lip}(X_{2\pi}; r; \gamma)$, $0 < \gamma \leqslant r$, be defined by

$$\mathrm{Lip}(X_{2\pi}; r; \gamma) := \{f \in X_{2\pi};\, \omega_r(X_{2\pi}; f; t) = \mathcal{O}(t^\gamma),\, t \to 0+\}.$$

Again it follows that $K(X_{2\pi}, X^r_{2\pi}; f, t)$ is equivalent to $\omega_r(X_{2\pi}; f; t^{1/r})$, so that $(X_{2\pi})_\omega$ for $\omega(t) := t^{\gamma/r}$ is characterized as $\mathrm{Lip}(X_{2\pi}; r; \gamma)$ (cf. [4, p. 258]). In view of $\omega(t) := t^{\gamma/r}$, $\varphi(\rho) = \rho^{-r}$ (cf. (6.4)) one has

$$\omega(\varphi(\rho))/\varphi(\rho) = \rho^{r-\gamma}. \tag{6.5}$$

Therefore (4.3) is satisfied for $0 < \gamma < r$. If $\gamma = r$, then in view of Proposition 4.1 and (6.4) condition (4.3) may be dispensed with if $\|S_\rho\|_{[X_{2\pi}]} = o(n^r)$; this is certainly the case if $r > (N-1)/2$ (cf. (6.3)). Hence by Theorem 5.1(a)

COROLLARY 6.3. *Let $\alpha > (N-1)\,|(1/p)-(1/2)|$, $r \in \mathbb{N}$, $0 < \gamma \leqslant r$, and suppose that $r > (N-1)/2$ if $\gamma = r$. Then for $\tau \in (L^{p'}_{2\pi})^\wedge$*

$$\|D^\tau_\rho\|_{p'} = o(\rho^\gamma) \Leftrightarrow \tau \in M(\mathrm{Lip}(L^p_{2\pi}; r; \gamma), (C_{2\pi})_0).$$

6.3. *Jacobi Expansions*

With $w(\theta) := (\sin(\theta/2))^{2a+1}(\cos(\theta/2))^{2b+1}$, $a \geqslant b \geqslant -1/2$, $a > -1/2$ let L^p_w, $1 \leqslant p \leqslant \infty$, be the space of functions for which

$$\|f\|_{p,w} := \left[\int_0^\pi |f(\theta)|^p\, w(\theta)\, d\theta\right]^{1/p}, \qquad 1 \leqslant p < \infty,$$
$$:= \operatorname*{ess\,sup}_{0\leqslant\theta\leqslant\pi} |f(\theta)|, \qquad p = \infty,$$

respectively, is finite. Let $C_w \subset L_w^\infty$ be the subspace of functions, continuous on $[0, \pi]$. If the Jacobi polynomial $\tilde{P}_k^{(a,b)}$, $k \in \mathbb{N}_0$, is given by

$$(1-u)^a(1+u)^b\, \tilde{P}_k^{(a,b)}(u) := \frac{(-1)^k}{2^k k!}\left(\frac{d}{du}\right)^k [(1-u)^{k+a}(1+u)^{k+b}]$$

and the modified one $P_k^{(a,b)}$ by

$$P_k^{(a,b)}(u) := \tilde{P}_k^{(a,b)}(u)/\tilde{P}_k^{(a,b)}(1), \qquad \tilde{P}_k^{(a,b)}(1) = \binom{k+a}{k},$$

then the sequence $\{f_k; k \in \mathbb{N}_0\}$,

$$f_k(\theta) := \left[\frac{1}{2k+a+b+1}\,\frac{\Gamma(k+a+1)\,\Gamma(k+b+1)}{\Gamma(k+1)\,\Gamma(k+a+b+1)}\right]^{-1/2} P_k^{(a,b)}(\cos\theta)$$
$$=: (h_k^{(a,b)})^{1/2}\, P_k^{(a,b)}(\cos\theta),$$

is orthonormal in L_w^2. It follows that the spaces C_w and L_w^p, $1 \leqslant p < \infty$, are admissible with respect to L_w^2, $\{f_k\}$. They are also regular, since (2.13) holds true for $\alpha > (2a+1)\,|(1/2)-(1/p)|$ (cf. [34, p. 258; 40, p. 88] and the literature cited there). Following Askey, Wainger and Gasper (cf. [11, Chap. III and the literature cited there]), one may introduce a (generalized) translation operator T_φ, $0 \leqslant \varphi \leqslant \pi$, via

$$f_k^*(T_\varphi f) := P_k^{(a,b)}(\cos\varphi) f_k^*(f) \qquad (k \in \mathbb{N}_0),$$

where $f_k^*(f) := \int_0^\pi f(\theta)\, f_k(\theta)\, w(\theta)\, d\theta$ is the kth Fourier–Jacobi coefficient. For $f \in C_w$ or $f \in L_w^p$, $1 \leqslant p < \infty$, one has

$$\|T_\varphi f\|_{p,w} \leqslant \|f\|_{p,w}, \qquad \lim_{\varphi \to 0+} \|T_\varphi f - f\|_{p,w} = 0.$$

Defining convolution of $f \in L_w^p$, $g \in L_w^1$ by

$$(f * g)(\theta) := \int_0^\pi (T_\varphi f)(\theta)\, g(\varphi)\, w(\varphi)\, d\varphi,$$

it has the familiar properties

$$\|f * g\|_{p,w} \leqslant \|f\|_{p,w} \|g\|_{1,w}, \qquad f_k^*(f * g) = [h_k^{(a,b)}]^{-1/2} f_k^*(f)\, f_k^*(g).$$

With Dirichlet kernel $D_n(\theta) := \sum_{k=0}^n (h_k^{(a,b)})^{1/2} f_k(\theta)$ one may write

$$(S_n f)(\theta) := \sum_{k=0}^n f_k^*(f)\, f_k(\theta) = (f * D_n)(\theta),$$

and it follows that (cf. [34, Sects. 9.1, 9.41])

$$\|S_n\|_{[L_w^p]} \leqslant \|D_n\|_{1,w} = \mathcal{O}(n^{a+(1/2)}) \qquad (n \to \infty), \tag{6.6}$$

the latter estimate being sharp for $p = 1$ (and $p = \infty$). For $f \in L_w^p$ one may introduce a (generalized) modulus of continuity by

$$\omega_p(f;\delta) := \sup_{0\leqslant\omega\leqslant\delta} \|T_\omega f - f\|_{p,w}$$

and corresponding Lipschitz spaces by $(0<\gamma\leqslant 2)$

$$\mathrm{Lip}(p,\gamma) := \{f\in L^p_w;\ \omega_p(f;\delta)=\mathcal{O}(\delta^\gamma),\ \delta\to 0+\}.$$

Setting

$$Z := \{g\in L^p_w;\text{ there exists } G\in L^p_w \text{ such that}$$
$$-k(k+a+b+1)\,f^*_k(g)=f^*_k(G)\text{ for all } k\in\mathbb{N}_0\},$$

with $|g|_Z := \|G\|_{p,w}$ it follows that (cf. [28])

$$C_1K(L^p_w, Z; f, t)\leqslant \omega_p(f;t^{1/2})\leqslant C_2K(L^p_w, Z; f, t).$$

Notice that with $\Delta := (1/w(\theta))(d/d\theta)[w(\theta)(d/d\theta)]$

$$\Delta P^{(a,b)}_k(\cos\theta) = -k(k+a+b+1)\,P^{(a,b)}_k(\cos\theta).$$

One has the Jackson-type inequality (cf. [5, 28, 40, p. 87])

$$E_n(g; L^p_w)\leqslant Cn^{-2}\,|g|_Z \tag{6.7}$$

and for polynomials $t_n(\theta) := \sum_{k=0}^n \alpha_k f_k(\theta)$ the Bernstein-type inequality (cf. [13, 40, p. 90])

$$|t_n|_Z = \|\Delta t_n\|_{p,w}\leqslant Cn^2\|t_n\|_{p,w}.$$

Again we only consider the case $\omega(t) := t^{\gamma/2}$, $0<\gamma\leqslant 2$, so that $(L^p_w)_\omega$ is characterized as $\mathrm{Lip}(p,\gamma)$. Since $\varphi(n)=n^{-2}$ (cf. (6.7)), one has $\omega(\varphi(n))/\varphi(n)=n^{2-\gamma}$ so that (4.3) is satisfied for $0<\gamma<2$. If $\gamma=2$, then in view of Proposition 4.1 and (6.7) condition (4.3) may indeed be avoided if $\|S_n\|_{[L^p_w]}=o(n^2)$ which is certainly the case if $a<3/2$ (cf. (6.6)). An application of Corollary 5.2, however, delivers

COROLLARY 6.4. *Let* $a\geqslant b\geqslant -1/2$, $a>-1/2$, $\alpha>(2a+1)|(1/2)-(1/p)|$, *and* $0<\gamma\leqslant 2$. *Then for* $\tau\in M(L^p_w, L^q_w)$, $1\leqslant p, q<\infty$,

$$\|\tau(n)\|_{M(L^p_w, L^q_w)}=\mathcal{O}(n^\gamma)\Leftrightarrow \tau\in M(\mathrm{Lip}(p,\gamma), (L^q_w)_h).$$

7. Concluding Remarks

A further group of results dealing with multipliers of uniform convergence on $C_{2\pi}$ is concerned with quasi-convex sequences, thus with sequences $\tau := \{\tau_k; k\in\mathbb{Z}\}$ for which $\tau_{-k}=\tau_k\in\mathbb{C}$ and

$$\sum_{k=0}^{\infty} (k+1)\,|\tau_k - 2\tau_{k+1} + \tau_{k+2}| < \infty.$$

Again at the end of a long development there is the following result of Teljakovskii [35].

THEOREM 7.1. *For quasi-convex sequences one has*

$$\omega(1/n)\,\tau_n \log n = o(1) \Leftrightarrow \tau \in M((C_{2\pi})_\omega, (C_{2\pi})_0).$$

Thus the intricate term $\|D_n^\tau\|_1$ in Theorem 1.1 is replaced by $\tau_n \log n$, but only for the subclass of quasi-convex sequences (every bounded quasi-convex sequence belongs to $M(C_{2\pi}, C_{2\pi})$, cf. [6, p. 249]). As is shown in [23], the following abstract counterpart is available.

THEOREM 7.2. *Let X be admissible (with respect to some orthonormal structure H, $\{f_k; k \in \mathbb{Z}\}$) such that* (2.13) *is satisfied for $\alpha = 1$. Suppose that the Jackson-type inequality* (3.11) *and the Bernstein-type inequality* (4.4) *hold true for some $Z \subset X$ and for ω, φ subject to* (4.3). *If τ is bounded and quasi-convex, then $(n \to \infty)$*

$$\omega(\varphi(n))\,\tau_n \|S_n\|_{[X]} = o(1) \Leftrightarrow \tau \in M(X_\omega, X_0).$$

Apart from an immediate recapture of Theorem 7.1 (cf. (6.1–6.2)), Theorem 7.2 now allows new applications to, e.g., Legendre, Laguerre, and Hermite expansions (see [23] for the details). Concerning expansions which satisfy (2.13) only for $\alpha > 1$, one may still establish counterparts of the sufficiency of Theorem 7.2 (cf. [21, 22]), but at present the necessity part remains open (see also [23a]).

Note added in proof. In the meantime, the gliding hump method as presented in Section 4 has indeed been extended to a uniform boundedness principle with rates. This has interesting applications in connection with the existence of counterexamples for direct estimates in approximation theory and numerical analysis. For details see W. Dickmeis and R. J. Nessel, A unified approach to certain counterexamples in approximation theory in connection with a uniform boundedness principle with rates, *J. Approx. Theory*, in press. In fact, the more general approach given there does not depend on any orthogonal structures or projection properties (like (4.18*)).

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8. References

1. S. Aljancic, Über Summierbarkeit von Orthogonalentwicklungen stetiger Funktionen, *Acad. Serbe Sci. Publ. Inst. Math.* **10** (1956), 121–130.
2. R. Bojanic, On uniform convergence of Fourier series, *Acad. Serbe Sci. Publ. Inst. Math.* **10** (1956), 153–158.
3. P. L. Butzer, The Banach–Steinhaus theorem with rates, and applications to various branches of analysis, *in* "General Inequalities II," Proc. Conf. Oberwolfach 1978 (E. F. Beckenbach, Ed.), ISNM 47, pp. 299–331, Birkhäuser, Basel, 1980.
4. P. L. Butzer and H. Berens, "Semi-Groups of Operators and Approximation," Springer-Verlag, Berlin, 1967.
5. P. L. Butzer and J. Junggeburth, On Jackson-type inequalities in approximation theory, *in* "General Inequalities I," Proc. Conf. Oberwolfach 1976 (E. F. Beckenbach, Ed.), ISNM 41, pp. 85–114, Birkhäuser, Basel, 1978.
6. P. L. Butzer and R. J. Nessel, "Fourier Analysis and Approximation," Birkhäuser, Basel, and Academic Press, New York, 1971.
7. P. L. Butzer, R. J. Nessel, and W. Trebels, On summation processes of Fourier expansions in Banach spaces, I: Comparison theorems; II: Saturation theorems; III: Jackson- and Zamansky-type inequalities for Abel-bounded expansions, *Tôhoku Math. J.* **24** (1972), 127–140; 551–569; **27** (1975), 213–223.
8. P. L. Butzer, R. J. Nessel, and W. Trebels, Multipliers with respect to spectral measures in Banach spaces and approximation, I: Radial multipliers in connection with Riesz-bounded spectral measures, *J. Approx. Theory* **8** (1973), 335–356.
9. P. L. Butzer and K. Scherer, Jackson and Bernstein-type inequalities for families of commutative operators in Banach spaces, *J. Approx. Theory* **5** (1972), 308–342.
10. R. DeVore, Multipliers of uniform convergence, *Enseignement Math.* **14** (1969), 175–188.
11. G. Gasper, "Special Functions," Seminar Lectures, RWTH Aachen, 1974/1975.
12. G. Goes, Multiplikatoren starker Konvergenz von Fourierreihen, I, II, *Studia Math.* **17** (1958), 299–308; 309–311.
13. E. Görlich, R. J. Nessel, and W. Trebels, Bernstein-type inequalities for families of multiplier operators in Banach spaces with Cesàro decompositions, I: General theory; II: Applications, *Acta Sci. Math. (Szeged)* **34** (1973), 121–130; **36** (1974), 39–48.
14. F. I. Harsiladse, Uniform convergence factors and uniform summability (Russ.), *Akad. Nauk Gruzin. SSR Trudy Tbiliss. Mat. Inst. Razmadze* **26** (1959), 121–130.
15. S. A. Husain, Convergence factors of Fourier series of summable functions, *J. Reine Angew. Math.* **259** (1973), 183–185.
16. B. A. Iljin, Problems of localization and convergence for Fourier series in fundamental systems of functions of the Laplace operator (Russ.), *Uspehi Mat. Nauk* **23** (1968), 61–120; *Russian Math. Surveys* **23** (1968), 59–116.
17. J. Karamata, Suite de fonctionelles lineaires et facteurs de convergence des séries de Fourier, *J. Math. Pures Appl.* **35** (1956), 87–95.
18. M. Katayama, Fourier series VII: Uniform convergence factors of Fourier series, *J. Fac. Sci. Hokkaido Univ. Ser. I* **13** (1957), 121–129.
19. T. Kojima, On generalized Toeplitz's theorems on limit and their applications, *Tôhoku Math. J.* **12** (1917), 291–326.
20. H. J. Mertens, Über Multiplikatoren starker Konvergenz, Dissertation, RWTH Aachen, 1979.
21. H. J. Mertens and R. J. Nessel, Über Multiplikatoren starker Konvergenz für Fourier–Entwicklungen in Banach–Räumen, *Math. Nachr.* **84** (1978), 185–194.
22. H. J. Mertens and R. J. Nessel, Ein hinreichendes Kriterium für α-fach quasikonvexe

Multiplikatoren starker Konvergenz, *in* "Approximation and Function Spaces," Proc. Conf. Gdańsk 1979 (Z. Ciesielski, Ed.), in press.
23. H. J. MERTENS AND R. J. NESSEL, Quasikonvexe Multiplikatoren starker Konvergenz, *Anal. Math.* **7** (1981), in press.
23a. H. J. MERTENS AND R. J. NESSEL, α-fold quasiconvex multipliers of strong convergence between different Banach spaces, *in* "Functions, Series, Operators," Proc. Conf. Budapest 1980, in press.
24. H. J. MERTENS, R. J. NESSEL, AND G. WILMES, Über Multiplikatoren zwischen verschiedenen Banach–Räumen im Zusammenhang mit diskreten Orthogonalentwicklungen, Forschungsberichte des Landes NRW No. 2599, Westdeutscher Verlag, Opladen, 1976.
25. R. J. NESSEL AND G. WILMES, On Nikolskii-type inequalities for orthogonal expansions, *in* "Approximation Theory II," Proc. Conf. Austin, Texas, 1976 (G. G. Lorentz, C. K. Chui, L. L. Schumaker, Eds.), pp. 479–484, Academic Press, New York, 1976.
26. R. J. NESSEL AND G. WILMES, Über Ungleichungen vom Bernstein–Nikolskii–Riesz-Typ in Banach–Räumen, Forschungsberichte des Landes NRW No. 2841, Westdeutscher Verlag, Opladen, 1979.
27. S. M. NIKOLSKII, "Approximation of Functions of Several Variables and Imbedding Theorems," Springer-Verlag, Berlin, 1975.
28. S. PAWELKE, Konstruktive Funktionentheorie mit Hilfe von Jacobi-Polynomen und Kugelfunktionen, *in* "Approximation Theory," Proc. Conf. Poznań 1972 (Z. Ciesielski, J. Musielak, Eds.), pp. 157–173, Polish, Sci. Publ., Warsaw, and Reidel, Boston, 1975.
29. V. R. POCHUEV, On multipliers of uniform convergence and multipliers of uniform boundedness of partial sums of Fourier series (Russ.), *Izv. Vyss. Ucebn. Zaved Matematika* **21** (1977), 74–81; *Soviet Math.* **21** (1977), 60–66.
30. H. S. Shapiro, Lebesgue constants for spherical partial sums, *J. Approx. Theory* **13** (1975), 40–44.
31. V. V. SHUK, Some modifications of the concept of modulus of smoothness and their applications (Russ.), *Dokl. Akad. Nauk SSSR* **162** (1965), 19–22; *Soviet Math. Dokl.* **6** (1965), 617–620.
32. E. M. STEIN, Interpolation in polynomial classes and Markoff's inequality, *Duke Math. J.* **24** (1957), 467–476.
33. E. M. STEIN AND G. WEISS, "Fourier Analysis on Euclidean Spaces," Princeton Univ. Press, Princeton, N.J., 1971.
34. G. SZEGÖ, "Orthogonal Polynomials," Amer. Math. Soc. Colloq. Publ. No. 23, Providence, R. I., 1959.
35. S. A. TELJAKOVSKII, Quasiconvex uniform convergence factors for Fourier series of functions with a given modulus of continuity (Russ.), *Mat. Zametki* **8** (1970), 619–623; *Math. Notes* **8** (1970), 817–819.
36. S. A. TELJAKOVSKII, Uniform convergence factors for Fourier series of functions with a given modulus of continuity (Russ.), *Mat. Zametki* **10** (1971), 33–40; *Math. Notes* **10** (1971), 444–448.
37. A. F. TIMAN, "Theory of Approximation of Functions of a Real Variable," Pergamon, Oxford, 1963.
38. M. TOMIC, Sur les facteurs de convergence des séries de Fourier des fonctions continues, *Acad. Serbe Sci. Publ. Inst. Math.* **8** (1955), 23–32.
39. M. TOMIC, Sur la sommation de la série de Fourier d'une fonction continue avec le module de continuité donné, *Acad. Serbe Sci. Publ. Inst. Math.* **10** (1956), 19–36.
40. W. TREBELS, "Multipliers for (C, α)-Bounded Fourier Expansions in Banach Spaces and Approximation Theory," Lecture Notes in Math. No. 329, Springer-Verlag, Berlin, 1973.